

# Superconductor coupled to two Luttinger liquids as an entangler for electron spins

Patrik Recher and Daniel Loss

*Department of Physics and Astronomy, University of Basel,  
Klingelbergstrasse 82, CH-4056 Basel, Switzerland*

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We consider an s-wave superconductor (SC) which is tunnel-coupled to two spatially separated Luttinger liquid (LL) leads. We demonstrate that such a setup acts as an entangler, i.e. it creates spin-singlets of two electrons which are spatially separated, thereby providing a source of electronic Einstein-Podolsky-Rosen pairs. We show that in the presence of a bias voltage, which is smaller than the energy gap in the SC, a stationary current of spin-entangled electrons can flow from the SC to the LL leads due to Andreev tunneling events. We discuss two competing transport channels for Cooper pairs to tunnel from the SC into the LL leads. On the one hand, the coherent tunneling of two electrons into the same LL lead is shown to be suppressed by strong LL correlations compared to single-electron tunneling into a LL. On the other hand, the tunneling of two spin-entangled electrons into different leads is suppressed by the initial spatial separation of the two electrons coming from the same Cooper pair. We show that the latter suppression depends crucially on the effective dimensionality of the SC. We identify a regime of experimental interest in which the separation of two spin-entangled electrons is favored. We determine the decay of the singlet state of two electrons injected into different leads caused by the LL correlations. Although the electron is not a proper quasiparticle of the LL, the spin information can still be transported via the spin density fluctuations produced by the injected spin-entangled electrons.

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## I. INTRODUCTION

Pairwise and non-local entangled quantum states, so-called Einstein-Podolsky-Rosen (EPR) pairs [1], represent the fundamental resource for quantum communication [2] schemes like dense coding, quantum teleportation or quantum key distribution [3], or more fundamentally, they can be used to test Bell's inequalities [4]. Experiments have tested Bell's inequalities [5], dense coding [6], and quantum teleportation [7, 8] using photons, but to date no experiments for *massive* particles like electrons in a solid state environment exist. This is so because it is difficult to first produce entangled electrons and also to detect them afterwards in a controlled way due to other electrons interacting with the entangled pair. On the other hand, the spin of an electron was pointed out to be a most natural candidate for a quantum bit (qubit) [9, 10]. This idea was supported also by experiments which show unusually long dephasing times for electron spins in semiconductors (approaching microseconds) and phase coherent transport up to  $100\mu\text{m}$  [11, 12, 13]. In addition, the electron also possesses charge which makes it well suited for transporting the spin information [14, 18]. Further, the Coulomb interaction between the electron charges can be exploited to spatially separate the spin-entangled electrons resulting in electronic EPR pairs. As first pointed out in Refs. [14, 18], such electronic EPR pairs can be used for testing Bell inequalities and for quantum communication schemes in the solid state. The first step towards this goal is to have a scheme by which the electrons can be reliably entangled. One possibility is to use coupled quantum dots [14, 18]. Alternatively, we

recently proposed an entangler device [16], which creates mobile and non-local spin-entangled electrons, consisting of an s-wave superconductor (SC), where the electrons are correlated in Cooper pairs with spin-singlet wavefunctions [15]. The SC is tunnel-coupled via two quantum dots in the Coulomb blockage regime [17] to two spatially separated Fermi liquid leads. By applying a bias, a stationary current of spin-entangled electrons can flow from the SC to the leads. The quantum dots are used to mediate the necessary interaction between the two electrons initially forming a Cooper pair in the SC so that the two electrons tunnel preferably not into the same but instead into different leads. This entangler then satisfies all requirements to detect the entanglement via the current noise in a beam splitter setup [18]. It is straightforward to formulate spin measurements for testing Bell inequalities (it is most promising to measure spin via charge [9, 19, 20]). We refer to related work [21, 22, 23], which makes also use of Andreev tunneling, but in a regime opposite to the one considered in Ref. [16] and here, where the superconductor/normal interface is transparent and no Coulomb blockade nor strong correlations are present.

In the present work we propose and discuss an alternative realization of an entangler which is based on strongly interacting one-dimensional wires which show Luttinger liquid (LL) behavior. In comparison to our earlier proposal with quantum dots [16], we replace now the Coulomb blockade behavior of the dots by strong correlations of the LL. Well-known examples for LL candidates are carbon nanotubes [24]. The low energy excitations of these LL are collective charge and spin modes rather than quasiparticles which resemble free electrons like they exist in a Fermi liquid. As a consequence,

the single-electron tunneling into a LL is suppressed by strong correlations. The question then arises quite naturally whether these strong correlations can even further suppress the coherent tunneling of *two* electrons into the same LL, as provided by a correlated two-particle tunneling event (Andreev tunneling), so that the two electrons preferably separate and tunnel into different LL leads. It turns out that the answer is positive. To address this question we introduce a setup consisting of an s-wave SC which is weakly tunnel-coupled to the center (bulk) of two spatially separated one-dimensional wires 1,2 described as Luttinger liquids, see Fig. 1,2. In this model we calculate the stationary current generated by the tunneling of a singlet (spin-entangled electrons), transferred from the SC into two separate leads (non-local process) or into the same lead (local process), 1 or 2. We show that the ratio of these two competing current channels depends on the system parameters and that it can be made large in order to have the desired injection of the two electrons in two separate leads, where, again, the two spins, forming a singlet, are entangled in spin space while separated in orbital space and therefore represent an electronic EPR pair. It is well-known that tunneling of single

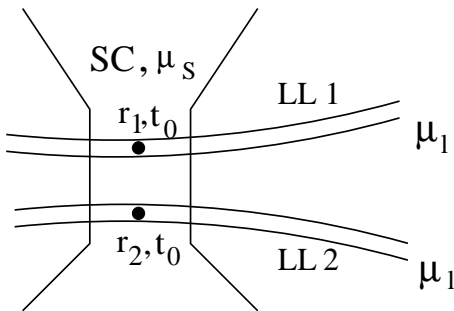


FIG. 1: A possible implementation of the entangler setup: Two quantum wires 1,2, described as infinitely long Luttinger liquids (LL), are deposited on top of an s-wave superconductor (SC) with chemical potential  $\mu_S$ . The electrons of a Cooper pair can tunnel by means of an Andreev process from two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  on the SC to the center (bulk) of the two quantum wires 1 and 2 resp. with tunneling amplitude  $t_0$ . The interaction between the leads is assumed to be negligible.

electrons into LLs is suppressed compared to Fermi liquids due to strong many-body correlation. In addition, we find now that subsequent tunneling of a second electron into the same LL is further suppressed, again in a characteristic interaction dependent power law, provided the applied voltage bias between the SC and the LL is much smaller than the energy gap  $\Delta$  in the SC so that single-electron tunneling is suppressed. The two-particle tunneling event is strongly correlated within the uncertainty time  $\hbar/\Delta$ , characterizing the time-delay between subsequent tunneling events of the two electrons of the same Cooper pair. In other words, the second electron of a Cooper pair is influenced by the existence of its preceding partner electron already present in the LL. This

effect can also be interpreted as a Coulomb blockade effect, similar to what occurs in quantum dots attached to a SC [16, 50]. Similar Coulomb blockade effects occur also in a mesoscopic chiral LL within a quantum dot coupled to macroscopic chiral LL edge-states in the fractional quantum Hall regime [25]. There, the Coulomb blockade-like energy gap is quantized in units of the non-interacting energy level spacing of the quantum dot and its existence is therefore a finite size effect, whereas in the present case, we will see that the suppression comes from strong correlations in a two-particle tunneling event which is present even in an infinitely long LL as considered here. On the other hand, if the two electrons of a Cooper pair tunnel to different leads, they will preferably tunnel from different points from the SC  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , with distance  $\delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  due to the spatial separation of the leads, see Fig. 1,2. We find that the current is exponentially suppressed if the distance  $\delta r$  exceeds the coherence length  $\xi$  of a Cooper pair on the SC. This limitation poses no severe experimental restriction since  $\xi$  is on the order of micrometers for usual s-wave materials, and  $\delta r$  can be assumed to be on the order of nanometers. Still, a power law suppression  $\propto 1/(k_F \delta r)^2$ , with  $k_F$  being the Fermi wavevector in the SC, remains and is more relevant. We show, however, that in lower dimensions of

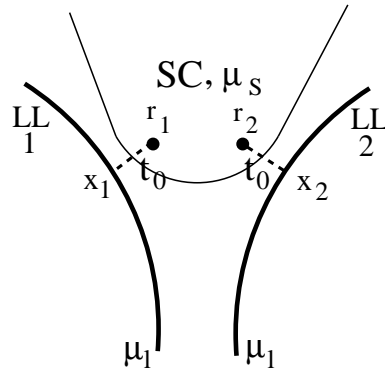


FIG. 2: An alternative implementation of the proposed entangler set-up: The two quantum wires 1,2 with chemical potential  $\mu_l$ , described as infinitely long Luttinger liquids (LL), are tunnel-coupled with amplitude  $t_0$  from two points  $x_1$  and  $x_2$  to two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  of a superconducting (SC) tip with chemical potential  $\mu_S$ .

the SC this suppression is less pronounced (with smaller powers). Further, we then discuss the decay in time of a spin-singlet state injected into two LL, one electron in each lead, due to the interaction present in the LL and find a characteristic power law decay in time at zero temperature. Despite this decay of the singlet state, the spin information can still be transported through the LL wires via the spin-density fluctuations created by the injected electrons.

## II. MODEL AND HAMILTONIAN

We consider an s-wave superconductor (SC) which is weakly tunnel-coupled to the center (bulk) of two spatially separated Luttinger liquid (LL) leads (see Fig. 1,2). The Hamiltonian of the whole system is represented as  $H = H_0 + H_T$  with  $H_0 = H_S + \sum_{n=1,2} H_{Ln}$  describing the isolated SC and LL-leads 1,2 respectively. Tunneling between the SC and the leads is governed by the tunneling Hamiltonian  $H_T$ . Each part of the system will be described in the following.

The s-wave SC with chemical potential  $\mu_S$  is described by the BCS-Hamiltonian[26]

$$H_S - \mu_S N_S = \sum_{\mathbf{k},s} E_{\mathbf{k}} \gamma_{\mathbf{k}s}^\dagger \gamma_{\mathbf{k}s}, \quad (1)$$

where  $s = (\uparrow, \downarrow)$  and  $N_S = \sum_{\mathbf{k}s} c_{\mathbf{k}s}^\dagger c_{\mathbf{k}s}$  is the number operator for electrons in the SC. The quasiparticle operators  $\gamma_{\mathbf{k}s}$  describe excitations out of the BCS-groundstate  $|0\rangle_S$  defined by  $\gamma_{\mathbf{k}s}|0\rangle_S = 0$ . They are related to the electron annihilation and creation operators  $c_{\mathbf{k}s}$  and  $c_{\mathbf{k}s}^\dagger$  through the Bogoliubov transformation [26]

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow}^\dagger \\ c_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma_{-\mathbf{k}\downarrow} - v_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow}^\dagger, \end{aligned} \quad (2)$$

where  $u_{\mathbf{k}} = (1/\sqrt{2})(1 + \xi_{\mathbf{k}}/E_{\mathbf{k}})^{1/2}$  and  $v_{\mathbf{k}} = (1/\sqrt{2})(1 - \xi_{\mathbf{k}}/E_{\mathbf{k}})^{1/2}$  are the usual BCS coherence factors [26], and  $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu_S$  is the normal state single-electron energy counted from the Fermi level  $\mu_S$ , and  $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$  is the quasiparticle energy. The field operator for an electron with spin  $s$  is  $\Psi_s(\mathbf{r}) = V^{-1/2} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} c_{\mathbf{k}s}$ , where  $V$  is the volume of the SC.

The two leads 1,2 are supposed to be infinite one dimensional interacting electron systems described by LL-theory. We only include forward scattering processes which describe scattering of electrons on the same branch (left or right movers). We neglect backscattering interactions which involve large momentum transfers of order  $2p_F$  where  $p_F$  is the Fermi wavevector in the LL [27]. The LL-Hamiltonian for the low energy excitations of lead  $n = 1, 2$  can then be written in a bosonized form as [28]

$$H_{Ln} - \mu_l N_n = \sum_{\nu=\rho,\sigma} \int_{-L/2}^{+L/2} dx \left( \frac{\pi u_\nu K_\nu}{2} \Pi_{n\nu}^2 + \frac{u_\nu}{2\pi K_\nu} (\partial_x \phi_{n\nu})^2 \right), \quad (3)$$

where the fields  $\Pi_n(x)$  and  $\phi_n(x)$  satisfy bosonic commutation relations  $[\phi_{n\nu}(x), \Pi_{m\mu}(x')] = i\delta_{nm}\delta_{\nu\mu}\delta(x - x')$ , and  $\mu_l$  is the chemical potential of the LL-leads (assumed to be identical for both leads), and  $N_n =$

$\sum_s \int dx \psi_{ns}^\dagger(x) \psi_{ns}(x)$  is the number operator for electrons in LL  $n$ . The Hamiltonian (3) describes long-wavelength charge ( $\nu = \rho$ ) and spin ( $\nu = \sigma$ ) density oscillations in the LL propagating with velocities  $u_\rho$  and  $u_\sigma$ , respectively. The velocities  $u_\nu$  and the stiffness parameters  $K_\nu$  depend on the interactions between the electrons in the LL. In the limit of vanishing backscattering, we have  $u_\sigma = v_F$  and  $K_\sigma = 1$ , and the LL is described by only two parameters  $K_\rho < 1$  and  $u_\rho$ . In a system with full translational invariance we have  $u_\rho = v_F/K_\rho$ . We decompose the field operator describing electrons with spin  $s$  into a right and left moving part,  $\psi_{ns}(x) = e^{ip_F x} \psi_{ns+}(x) + e^{-ip_F x} \psi_{ns-}(x)$ . The right (left) moving field operator  $\psi_{ns+}(x)$  ( $\psi_{ns-}(x)$ ) is then expressed as an exponential of bosonic fields as [29, 30]

$$\begin{aligned} \psi_{ns\pm}(x) &= \lim_{\alpha \rightarrow 0} \frac{\eta_{\pm,ns}}{\sqrt{2\pi\alpha}} \exp \left\{ \pm \frac{i}{\sqrt{2}} \left( \phi_{n\rho}(x) \right. \right. \\ &\quad \left. \left. + s\phi_{n\sigma}(x) \mp (\theta_{n\rho}(x) + s\theta_{n\sigma}(x)) \right) \right\}, \end{aligned} \quad (4)$$

where  $[\phi_{n\nu}(x), \theta_{m\mu}(x')] = -i(\pi/2)\delta_{nm}\delta_{\nu\mu}\text{sgn}(x - x')$  and therefore  $\partial_x \theta_{n\nu}(x) = \pi \Pi_{n\nu}(x)$ . The operators  $\eta_{\pm,ns}$  are needed to ensure the correct fermionic anticommutation relations. In the thermodynamic (TD) limit ( $L \rightarrow \infty$ ),  $\eta_{\pm,ns}$  can be presented by Hermitian operators satisfying the anticommutation relation [28]  $\{\eta_r, \eta_{r'}\} = 2\delta_{rr'}$ , with  $r = \pm, ns$ . We adopt the convention throughout the paper that  $s = +1$  for  $s = \uparrow$ , and  $s = -1$  for  $s = \downarrow$ , if  $s$  has not the meaning of an operator index.

Transfer of electrons from the SC to the LL-leads is described by the tunneling Hamiltonian  $H_T = \sum_n H_{Tn} + \text{H.c.}$ , where  $H_{Tn}$  is defined as  $H_{Tn} = t_0 \sum_s \psi_{ns}^\dagger \Psi_s(\mathbf{r}_n)$ . The field operator  $\Psi_s(\mathbf{r}_n)$  annihilates an electron with spin  $s$  at point  $\mathbf{r}_n$  on the SC, and  $\psi_{ns}^\dagger$  creates it again with amplitude  $t_0$  at point  $x_n$  in the LL  $n$  which is nearest to  $\mathbf{r}_n$ , see Fig. 1,2. We assume that the spin is conserved during the tunneling process, and thus the tunneling amplitudes  $t_0$  do not depend on spin, and, for simplicity, are the same for both leads  $n = 1, 2$ . We remark that our point-contact approach for describing the electron transfer from the SC to the LL is the simplest possible description but it captures presumably the relevant features of a real device. The scheme shown in Fig. 2 has a geometry which suggests that electrons tunnel from point  $\mathbf{r}_n \rightarrow x_n$  which are closest to each other, due to the fact that  $t_0$  depends exponentially on the tunneling distance. In the setup shown in Fig. 1, a point-like tunnel contact between the SC and the LL might be induced by slightly bending the quantum wires (e.g. nanotubes). If the contact area has a finite extension, we note that the two electrons preferably tunnel from the same point on the SC, when they tunnel into the same lead, since the two-particle tunneling event is coherent and shows a suppression in the probability already on a length scale given by  $1/k_F$ , as we discuss in detail below.

### III. STATIONARY CURRENT FROM THE SC TO THE LL-LEADS

We now calculate the current of singlets, i.e. pairwise spin-entangled electrons (Cooper pairs), from the SC to the LL-leads due to Andreev tunneling [31, 32] in first non-vanishing order, starting from a general T-matrix approach [33]. We thereby distinguish two transport channels. First we calculate the current when two electrons tunnel from different points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  of the SC into *different* interacting LL-leads which are separated in space such that there is no inter-lead interaction. In this case the only correlation in the tunneling process is due to the superconducting pairing of electrons which results in a coherent two-electron tunneling process of opposite spins from different points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  of the SC, and with a delay time  $\sim \hbar/\Delta$  between the two tunneling events. Since the total spin is a conserved quantity  $[H, \mathbf{S}^2] = 0$ , the spin entanglement of a Cooper pair is transported to *different* LL-leads, thus leading to non-local spin-entanglement. On the other hand, if two electrons tunnel from the same point of the SC into the *same* LL-lead there is an additional correlation in the LL-lead itself due to the intra-lead interaction. It is the goal of this work to investigate how the transport current for tunneling of two electrons from the SC into the *same* LL-lead is effected by this additional correlation.

### IV. THE T-MATRIX

We apply a T-matrix (transmission matrix) approach [33] to calculate the current [16]. The stationary current of *two* electrons passing from the SC to the leads is then given by

$$I = 2e \sum_{f,i} W_{fi} \rho_i. \quad (5)$$

Here,  $W_{fi}$  is the transition rate from the superconductor to the leads, given by  $W_{fi} = 2\pi |\langle f | T(\varepsilon_i) | i \rangle|^2 \delta(\varepsilon_f - \varepsilon_i)$ . Here,  $T(\varepsilon_i) = H_T \frac{1}{\varepsilon_i + i\eta - H} (\varepsilon_i - H_0)$  is the on-shell transmission or T-matrix, with  $\eta$  being a positive infinitesimal which we set to zero at the end of the calculation. The T-matrix can be expanded in a power series in the tunneling Hamiltonian  $H_T$ ,

$$T(\varepsilon_i) = H_T + H_T \sum_{n=1}^{\infty} \left( \frac{1}{\varepsilon_i + i\eta - H_0} H_T \right)^n, \quad (6)$$

where  $\varepsilon_i$  is the energy of the initial state  $|i\rangle$ , which, in our case, is the energy of a Cooper pair at the Fermi surface of the SC,  $\varepsilon_i = 2\mu_S$ . Finally,  $\rho_i = \langle i | \rho | i \rangle$  is the stationary occupation probability for the entire system to be in the state  $|i\rangle$ . We work in the regime  $\Delta > \mu > k_B T$ , where  $\mu = \mu_S - \mu_L$  is the applied voltage bias between the SC and the leads, and  $T$  the

temperature with  $k_B$  the Boltzmann constant. The regime  $\Delta > \mu$  ensures that single electron tunneling from the SC to the leads is excluded and only tunneling of *two* coherent electrons of opposite spins is allowed. In the regime  $\mu > k_B T$  we only have transport from the SC to the leads, and not in the opposite direction. Since temperature is assumed to be the smallest energy scale in the system, we assume  $k_B T = 0$  in the calculation. The set of initial states  $|i\rangle$ , virtual states  $|v\rangle$  and final states  $|f\rangle$  consists of the BCS groundstate (GS)  $|0\rangle_S$  and excitations  $\gamma_{\mathbf{k}s}^\dagger |0\rangle_S$  for the SC and a complete set [29, 30] of energy eigenstates  $|N_{nr\nu}, \{b_{n\nu}\}\rangle$  of the LL-Hamiltonian  $H_{Ln}$  given in (3).  $N_{nr\nu}$  is the number of excess spin ( $\nu = \sigma$ ) and charge ( $\nu = \rho$ ) in branch  $r$  relative to the state where all single-particle states are filled up to the chemical potential  $\mu_L$ . The Bose operators  $b_{n\nu}$  form a continuous spectrum describing collective spin and charge modes and will be introduced in (10) and (11). The GS of the LL is then  $|0, 0\rangle$ , which means that we have a filled Fermi sea and no bosonic excitations. The energy contribution of the excess charge and spin is included in the so-called zero mode ( $k = 0$ ) terms in the diagonalized Hamiltonian  $K_{Ln}$  (12) and are of no importance in the TD-limit ( $L \rightarrow \infty$ ) considered here, since the contribution of these terms due to an additional electron on top of the GS is  $\mathcal{O}(1/L)$  and is neglected in (12). For a detailed description of the LL-Hamiltonian (3) including the zero-modes see Appendix A. Since we want to calculate the transition rate for transport of a Cooper pair to the leads, the final states  $|f\rangle$  of interest contain two additional electrons of opposite spins in the leads compared to the initial state  $|i\rangle$ .

### V. CURRENT $I_1$ FOR TUNNELING OF TWO ELECTRONS INTO DIFFERENT LEADS

We first calculate the current for tunneling of two spin-entangled electrons into different leads. We expand the T-matrix to second order in  $H_T$  and go over to the interaction representation by using  $\delta(\epsilon) = (1/2\pi) \int_{-\infty}^{+\infty} dt e^{i\epsilon t}$ , and  $\langle v | (\epsilon_i - H_0 + i\eta)^{-1} | v \rangle = -i \int_0^\infty dt e^{i(\epsilon_i - \epsilon_v + i\eta)t}$ . By transforming the time dependent phases into a time dependence of the tunneling Hamiltonian we can integrate out all final and virtual states. The forward current  $I_1$  for tunneling of two electrons into different leads can then be written as

$$I_1 = 2e \lim_{\eta \rightarrow 0} \sum_{\substack{n \neq n' \\ m \neq m'}} \int_{-\infty}^{\infty} dt \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} e^{i(2t-t'-t'')\mu} \langle H_{Tm}^\dagger(t-t'') H_{Tm'}^\dagger(t) H_{Tn}(t') H_{Tn'} \rangle, \quad (7)$$

where  $\langle \dots \rangle$  denotes  $\text{Tr} \rho \{ \dots \}$ . The bias has been introduced in a standard way [34], and the time dependence of the operators in (7) is then governed by  $H_{Tn}(t) = e^{i(K_{Ln}+K_S)t} H_{Tn} e^{-i(K_{Ln}+K_S)t}$  with  $K_{Ln} + K_S = H_{Ln} + H_S - \mu_L N_n - \mu_S N_S$ . The transport process involves two electrons of *different* spins which suggests that the average in (7) is of the form (suppressing time variables),  $\langle \dots \rangle = \sum_{ss'} \langle H_{Tm-s}^\dagger H_{Tm's'}^\dagger H_{Tn-s} H_{Tn's} \rangle$ , where  $H_{Tns}$  describes tunneling of spin  $s$  governed by  $H_{Tn}$ . The time sequence in (7) contains the dynamics of the hopping of a Cooper pair from the SC to the LL-leads (one electron per lead) and back. The times  $t'$  and  $t''$  are delay times between subsequent hoppings of two electrons from the *same* Cooper pair, whereas  $t$  is the time between injecting and taking out a Cooper pair. We evaluate the thermal average in (7) at zero temperature where the expectation value is to be taken in the groundstate of  $K_0$  which is the BCS groundstate of the SC and the bosonic vacuum of the LL-leads for a filled Fermi sea. We remark that since the interaction between the different subsystems ( $SC, L_1, L_2$ ) is included in the tunneling-perturbation, the expectation value factorizes into a SC-part times a LL-part. In addition, the LL-correlation function factorizes into two single-particle correlation functions due to the negligible interaction between the LL-leads 1, 2 (this will be not the case if two electrons tunnel into the *same* lead). Note that in the TD-limit the time dynamics of all LL-correlation functions will be governed by a Hamiltonian that depends only on Bose operators (see (12)). The operators  $\eta_{\pm,rs}$  commute with all Bose operators, and as a consequence  $\eta_{\pm,rs}$  are time independent. Therefore, interaction terms of the LL of the form  $\psi_\alpha \psi_\beta \psi_\gamma^\dagger \psi_\delta^\dagger$  can be written as  $\eta_\alpha \eta_\beta \eta_\gamma \eta_\delta \times (\text{Bose operators})$ , where  $\alpha, \beta, \gamma, \delta$  are composite indices containing  $r = \pm, ns$ . The correlation function in (7) is then of the form

$$\begin{aligned} & \sum_{\substack{n \neq n' \\ m \neq m'}} \langle H_{Tm}^\dagger(t-t'') H_{Tm'}^\dagger(t) H_{Tn}(t') H_{Tn'} \rangle \\ &= |t_0|^4 \sum_{\substack{s \\ n \neq m}} \langle \psi_{ns}(t-t'') \psi_{ns}^\dagger \rangle \langle \psi_{m-s}(t-t') \psi_{m-s}^\dagger \rangle \\ & \times \langle \Psi_s^\dagger(\mathbf{r}_n, t-t'') \Psi_{-s}^\dagger(\mathbf{r}_m, t) \Psi_{-s}(\mathbf{r}_m, t') \Psi_s(\mathbf{r}_n) \rangle \\ & - |t_0|^4 \sum_{\substack{s \\ n \neq m}} \langle \psi_{m-s}(t-t'-t'') \psi_{m-s}^\dagger \rangle \langle \psi_{ns}(t) \psi_{ns}^\dagger \rangle \\ & \times \langle \Psi_{-s}^\dagger(\mathbf{r}_m, t-t'') \Psi_s^\dagger(\mathbf{r}_n, t) \Psi_{-s}(\mathbf{r}_m, t') \Psi_s(\mathbf{r}_n) \rangle. \end{aligned} \quad (8)$$

The 4-point correlation functions of the SC can be calculated by Fourier decomposing  $\Psi_s(\mathbf{r}_n, t) = V^{-1/2} \sum_{\mathbf{k}} (u_{\mathbf{k}s} \gamma_{\mathbf{k}s} e^{-iE_{\mathbf{k}}t} + v_{\mathbf{k}s} \gamma_{-\mathbf{k}-s}^\dagger e^{iE_{\mathbf{k}}t}) e^{i\mathbf{k}\mathbf{r}_n}$ , with  $u_{\mathbf{k}s} = u_{\mathbf{k}}$ , and  $v_{\mathbf{k}\uparrow} = -v_{\mathbf{k}\downarrow} = v_{\mathbf{k}}$ . For the first correlation function

in (8) we then obtain

$$\begin{aligned} & V^2 \langle \Psi_s^\dagger(\mathbf{r}_n, t-t'') \Psi_{-s}^\dagger(\mathbf{r}_m, t) \Psi_{-s}(\mathbf{r}_m, t') \Psi_s(\mathbf{r}_n) \rangle \\ &= \sum_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} e^{-iE_{\mathbf{k}}t'} e^{iE_{\mathbf{k}'}t''} e^{i(\mathbf{k}+\mathbf{k}')\delta\mathbf{r}} \\ &+ \sum_{\mathbf{k}\mathbf{k}'} (v_{\mathbf{k}} v_{\mathbf{k}'})^2 e^{-iE_{\mathbf{k}}(t-t'')} e^{-iE_{\mathbf{k}'}(t-t')}, \end{aligned} \quad (9)$$

where  $\delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  is the distance vector between the two tunneling points in the SC. The first sum in (9) describes the (time-dependent) correlation of creating and annihilating a quasiparticle (with same spin), whereas the second term in (9) describes correlation of creating *two* quasiparticles (with different spin). It is obvious that the second term describes processes which involve final states  $|f\rangle$  in the T-matrix element  $\langle f|T(\varepsilon_i)|i\rangle$  that contain two excitations in the SC and, therefore, does not describe an Andreev process. In the regime  $\Delta > \mu$  such a process is not allowed by energy conservation. We will see this explicitly by calculating the integral over  $t$  which originates from the Fourier representation of the  $\delta$ -function present in the rate  $W_{fi}$ . Similarly, for the correlator  $\langle \Psi_{-s}^\dagger(\mathbf{r}_m, t-t'') \Psi_s^\dagger(\mathbf{r}_n, t) \Psi_{-s}(\mathbf{r}_m, t') \Psi_s(\mathbf{r}_n) \rangle$  in (8) we obtain (9) with a minus sign, and we have to replace  $t-t''$  by  $t-t'-t''$ , and  $t-t'$  by  $t$ , in the second term of (9).

To evaluate the LL-correlation functions in (8) we decompose the phase fields  $\phi_{n\nu}(x, t)$  and  $\theta_{n\nu}(x, t)$  into a sum over the spin and charge bosons (see also Appendix A),

$$\begin{aligned} \theta_{n\nu}(x, t) &= - \sum_p \text{sgn}(p) \sqrt{\frac{\pi}{2LK_\nu|p|}} e^{ipx} e^{-\alpha|p|/2} \\ &\times (b_{n\nu p} e^{-iu_\nu|p|t} - b_{n\nu-p}^\dagger e^{iu_\nu|p|t}), \end{aligned} \quad (10)$$

and

$$\begin{aligned} \phi_{n\nu}(x, t) &= \sum_p \sqrt{\frac{\pi K_\nu}{2L|p|}} e^{ipx} e^{-\alpha|p|/2} \\ &\times (b_{n\nu p} e^{-iu_\nu|p|t} + b_{n\nu-p}^\dagger e^{iu_\nu|p|t}). \end{aligned} \quad (11)$$

The spin and charge bosons satisfy Bose-commutation relations, in particular  $[b_{n\nu p}, b_{n'\nu'p'}^\dagger] = \delta_{rr'}$ , where  $r \equiv n\nu p$ , and the LL-groundstate is defined as  $b_{n\nu p}|0\rangle_{LL} = 0$ . The Hamiltonian (3) can then be written in terms of the  $b$ -operators as (see Appendix A)

$$K_{Ln} = \sum_{\nu p} u_\nu |p| b_{n\nu p}^\dagger b_{n\nu p}, \quad (12)$$

where we have subtracted the zero-point energy coming from the filled Dirac sea of negative-energy particle states. In all  $p$ -sums we will explicitly exclude  $p = 0$  as discussed in Section IV and is explained in more detail in

Appendix A. To account for the  $p$ -dependence of the interaction, we apply a high momentum-transfer cut-off  $\Lambda$  on the order of  $1/p_F$  so that  $K_\nu(p) = K_\nu$ ,  $u_\nu(p) = u_\nu$  for  $|p| < 1/\Lambda$  and  $K_\nu(p) = 1$ ,  $u_\nu(p) = v_F$  for  $|p| > 1/\Lambda$ . By writing  $\psi_{n sr}(x, t) = (2\pi\alpha)^{-1/2} \eta_{r, ns} e^{i\Phi_{n sr}(x, t)}$  with  $r = \pm$  and  $\Phi_{n sr}$  defined according to (4), we can represent the single-particle LL-correlation function as  $\langle \psi_{n sr}(x, t) \psi_{n sr}^\dagger \rangle = (2\pi\alpha)^{-1} \exp\{\langle \Phi_{n sr}(x, t) \Phi_{n sr} - (\Phi_{n sr}^2(x, t) + \Phi_{n rs}^2)/2 \rangle\}$  with the well-known result [35, 36, 37, 38]

$$\begin{aligned} G_{n rs}^1(x, t) &\equiv \langle \psi_{n sr}(x, t) \psi_{n sr}^\dagger \rangle \\ &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 0} \frac{\Lambda + i(v_F t - rx)}{\alpha + i(v_F t - rx)} \\ &\times \prod_{\nu=\rho, \sigma} \frac{1}{\sqrt{\Lambda + i(u_\nu t - rx)}} \left[ \frac{\Lambda^2}{(\Lambda + iu_\nu t)^2 + x^2} \right]^{\gamma_\nu/2} \end{aligned} \quad (13)$$

where  $\gamma_\nu = (K_\nu + K_\nu^{-1})/4 - 1/2 > 0$  is an interaction dependent parameter which describes the power law decay of the long time and long distance correlations. The factor in the first line of (13) is only important if one is interested in  $x, t$  satisfying  $|v_F t - rx| < \Lambda$  and is a result of including the  $p$ -dependence of the interaction parameters  $K_\nu$  and  $u_\nu$ . The LL-correlation function (13) has singular points as a function of time in the upper complex plane. It is now clear that the integration over  $t$  is only nonzero if the phase of  $e^{i\omega t}$  in (7) is positive, i.e.  $\omega > 0$ . Since the phases of the terms containing  $(v_{\mathbf{k}})^2$  in (9) depend also on  $t$ , the requirement for a nonzero contribution to the current from these terms requires  $2\mu - E_{\mathbf{k}} - E_{\mathbf{k}'} > 0$ . This, however, is excluded in our regime of interest, since  $E_{\mathbf{k}} + E_{\mathbf{k}'} \geq 2\Delta$ , and therefore, we will not consider these terms any further in what follows. We remark that for  $r \neq r'$ , the correlation function  $\langle \psi_{n sr}(x, t) \psi_{n sr'}^\dagger \rangle$  gives a negligible contribution in the TD-limit. This statement is true also for a finite size LL as long as the interaction preserves the total number of right- and left-movers. In our model the electrons tunnel into the same point  $x_n$  in LL  $n$ , i.e.  $x = 0$  in (13). In addition, LL 1 and 2 are assumed to be similar. We therefore have  $G_{n rs}^1(x, t) \equiv G^1(t)$ , and the current  $I_1$  can then be written as

$$\begin{aligned} I_1 &= (32e |t_0|^4 / V^2) \\ &\times \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} dt \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} e^{i(2t-t'-t'')\mu} \\ &\times \sum_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} e^{-iE_{\mathbf{k}} t'} e^{iE_{\mathbf{k}'} t''} e^{i(\mathbf{k}+\mathbf{k}')\delta\mathbf{r}} \\ &\times \{G^1(t-t'')G^1(t-t') + G^1(t-t'-t'')G^1(t)\}. \end{aligned} \quad (14)$$

We evaluate (14) in leading order in the small parameter  $\mu/\Delta$  and remark that the delay times  $t'$  and  $t''$  are restricted to  $t', t'' \lesssim 1/\Delta$ . This becomes clear if we set

$\delta\mathbf{r} = 0$  and express the contribution in (14) containing the dynamics of the SC as

$$\begin{aligned} &\sum_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} e^{-iE_{\mathbf{k}} t'} e^{iE_{\mathbf{k}'} t''} \\ &= (\pi\nu_S \Delta/2)^2 H_0^{(1)}(t''\Delta) H_0^{(2)}(t'\Delta), \end{aligned} \quad (15)$$

where  $H_0^{(1)}$  and  $H_0^{(2)}$  are Hankel functions of the first and second kind, and  $\nu_S$  is the energy DOS per spin in the SC at  $\mu_S$ . For times  $t', t'' > 1/\Delta$ , the Hankel functions are rapidly oscillating, since for large  $x$  we have  $H_0^{(1/2)}(x) \sim \sqrt{2/\pi x} \exp(\pm i(x - (\pi/4)))$ . In contrast, the time-dependent phase in (14) containing the bias  $\mu$  suppresses the integrand in (14) only for times  $|2t - t' - t''| > 1/\mu > 1/\Delta$ . Being interested only in the leading order in  $\mu/\Delta$ , we can assume that  $|t| > t', t''$  in the current formula (14), since the LL correlation functions are slowly decaying in time with the main contribution (in the integral) coming from large times  $|t|$ . In addition, since  $1/\mu > \Lambda/v_F$ , we can neglect the term containing the Fermi velocity  $v_F$  in (13). To test the validity of our approximations we first consider the non-interacting limit with  $K_\nu = 1$  and  $u_\nu = v_F$ , for which an analytic expression is also available for higher order terms in  $\mu/\Delta$ .

## VI. NON-INTERACTING LIMIT FOR CURRENT $I_1$

Let us first consider a 1D-Fermi liquid (i.e.  $K_\nu = 1$  and  $u_\nu = v_F$ ), and evaluate the integral over  $t$  in Eq. (14) in the non-interacting limit. The LL-correlation functions simplify to  $G^1(t) = (1/2\pi) \lim_{\alpha \rightarrow 0} 1/(\alpha + iv_F t)$ , and we are left with the integral

$$\begin{aligned} &\int_{-\infty}^{\infty} dt e^{i2t\mu} \{G^1(t-t'')G^1(t-t') + G^1(t-t'-t'')G^1(t)\} \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dt e^{i2t\mu} \left\{ \frac{1}{(\alpha + iv_F(t-t''))(\alpha + iv_F(t-t'))} \right. \\ &\quad \left. + \frac{1}{(\alpha + iv_F(t-t'-t''))(\alpha + iv_F t)} \right\} \Big|_{\alpha \rightarrow 0} \end{aligned} \quad (16)$$

which can be evaluated by closing the integration contour in the upper complex plane. Inserting then the result into Eq. (14) we get

$$\begin{aligned} I_1 &= (32e |t_0|^4 / V^2) \lim_{\eta \rightarrow 0} \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} \\ &\times \sum_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} e^{-iE_{\mathbf{k}} t'} e^{iE_{\mathbf{k}'} t''} e^{i(\mathbf{k}+\mathbf{k}')\delta\mathbf{r}} \\ &\times \frac{1}{\pi v_F^2} \left\{ \frac{\sin((t''-t')\mu)}{t''-t'} + \frac{\sin((t'+t'')\mu)}{t'+t''} \right\}. \end{aligned} \quad (17)$$

The sine-functions in (17) can be expanded in powers of  $\mu$ , and for  $\mu < \Delta$  it is sufficient to keep just the leading order term in  $\mu$  since the integrals over  $t', t''$  have the form

$$\int_0^\infty dt e^{-(\eta \pm iE_{\mathbf{k}})t} t^n = \frac{n!}{(\eta \pm iE_{\mathbf{k}})^{n+1}}, \quad (18)$$

where  $n = 1, 2, 3 \dots$ . Since  $E_{\mathbf{k}} \geq \Delta$ , higher powers in  $t', t''$  produce higher powers in  $\mu/\Delta$ , and, as expected, we can therefore ignore the dependence on  $t', t''$  in the LL-correlation functions. In contrast to this, when we consider the current for tunneling of two electrons into the same (interacting) LL-lead, we will see that the two-particle correlation function will not allow for such a simplification. In leading order in  $\mu/\Delta$ , the integrals over  $t', t''$  are evaluated according to (18) with  $n = 0$ , and we get an (effective) momentum-sum for the SC-correlations  $(\sum_{\mathbf{k}} \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{E_{\mathbf{k}}} \cos(\mathbf{k} \cdot \delta \mathbf{r}))^2$ . To evaluate this sum we use  $u_{\mathbf{k}} v_{\mathbf{k}} = \Delta/(2E_{\mathbf{k}})$  and linearize the spectrum around the Fermi-level  $\mu_S$ , since the Fermi energy in the SC  $\varepsilon_F > \Delta$ , with Fermi wavevector  $k_F$ . We then obtain ( $\delta r$  denotes  $|\delta \mathbf{r}|$ )

$$\sum_{\mathbf{k}} \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{E_{\mathbf{k}}} \cos(\mathbf{k} \cdot \delta \mathbf{r}) = \frac{\pi}{2} \nu_S \frac{\sin(k_F \delta r)}{k_F \delta r} e^{-(\delta r/\pi \xi)}. \quad (19)$$

In (19) we have introduced the coherence length of a Cooper pair in the SC,  $\xi = v_F/\pi \Delta$ . We finally obtain  $I_1^0$ , the current  $I_1$  in the non-interacting limit,

$$I_1^0 = e\pi\gamma^2\mu \left[ \frac{\sin(k_F \delta r)}{k_F \delta r} \right]^2 \exp\left(-\frac{2\delta r}{\pi \xi}\right). \quad (20)$$

Here we have defined  $\gamma = 4\pi\nu_S\nu_l|t_0|^2/LV$ , which is the dimensionless conductance per spin to tunnel from the SC to the LL-leads. The non-interacting DOS of the LL per spin  $\nu_l$  is given by  $\nu_l = L/\pi v_F$ . [We remark in passing that this result agrees with a  $T$ -matrix calculation in the energy domain [39]. In this case we sum explicitly over the final states, given by a singlet  $|f\rangle = (1/\sqrt{2})[a_{1p\uparrow}^\dagger a_{2q\downarrow}^\dagger - a_{1p\downarrow}^\dagger a_{2q\uparrow}^\dagger]|i\rangle$ , where the  $a$ -operators describe electrons in a non-interacting 1D Fermi-liquid. Note that triplet states are excluded as final states since our Hamiltonian  $H$  does not change the total spin.] We see that the current  $I_1^0$  gets exponentially suppressed on the scale of  $\xi$ , if the tunneling of the two (coherent) electrons takes place from different points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  of the SC. For conventional s-wave SC the coherence length  $\xi$  is typically on the order of micrometers and therefore this poses not severe experimental restrictions. Thus, in the regime of interest  $\delta r < \xi$ , the suppression of the current  $I_1^0$  is only polynomial, i.e.  $\propto (1/k_F \delta r)^2$ . It was shown [40] that a superconductor on top of a two-dimensional electron gas (2DEG) can induce superconductivity (by the proximity effect) in the 2DEG with a finite order

parameter. The 2DEG then becomes an effective two-dimensional (2D) SC. More recently, it was suggested that superconductivity should also be present in ropes of single-walled carbon nanotubes [41], which are one-dimensional (1D) systems. It is therefore interesting to calculate (19) also in 2D and 1D. In the case of a 2D SC we evaluate  $\sum_{\mathbf{k}} \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{E_{\mathbf{k}}} \cos(\mathbf{k} \cdot \delta \mathbf{r})$  in leading order in  $\delta r/\pi \xi$ , and we find

$$\sum_{\mathbf{k}(2D)} \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{E_{\mathbf{k}}} \cos(\mathbf{k} \cdot \delta \mathbf{r}) = \frac{\pi}{2} \nu_S \left( J_0(k_F \delta r) + 2 \sum_{\nu=1}^{\infty} \frac{J_{2\nu}(k_F \delta r)}{\pi \nu} \right), \quad (21)$$

where  $J_\nu(x)$  denotes the Bessel function of order  $\nu$ . For large  $k_F \delta r$ , we get  $J_\nu(k_F \delta r) \sim \sqrt{2/\pi k_F \delta r} \cos(k_F \delta r - (\nu\pi/2) - (\pi/4))$ , which allows an approximation of the right-hand side of (21) for large  $k_F \delta r$  by  $(\pi/2)\nu_S \sqrt{2/\pi k_F \delta r} \cos(k_F \delta r - (\pi/4))(1 - (2/\pi) \ln 2)$ . This result is exact to leading order in an expansion in  $1/k_F \delta r$ . So asymptotically, the current decays only  $\propto 1/k_F \delta r$ . For  $\delta r = 0$ , the bracket on the right-hand side of (21) becomes 1 as in the 3D-case.

In the case of a 1D-SC we obtain

$$\sum_{\mathbf{k}(1D)} \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{E_{\mathbf{k}}} \cos(\mathbf{k} \cdot \delta \mathbf{r}) = \frac{\pi}{2} \nu_S \cos(k_F \delta r) e^{-(\delta r/\pi \xi)}, \quad (22)$$

where there are only oscillations and no decay of the Andreev amplitude (for  $\delta r/\pi \xi < 1$ ). We see that the suppression of the current due to a finite separation of the tunneling points on the SC can be reduced considerably (or even excluded completely) by going over to lower dimensional SCs.

## VII. CURRENT $I_1$ INCLUDING INTERACTION.

We now are ready to treat the interacting case. Having obtained confidence in our approximation schemes from the non-interacting case above, we can neglect now the  $t', t''$  dependence of the LL-correlation function appearing in (14), valid in leading order in  $\mu/\Delta$ . In this limit the  $t$ -integral considerably simplifies to

$$\begin{aligned} & (2\pi)^2 \int_{-\infty}^{\infty} dt e^{i(2t-t'-t'')\mu} \\ & \times \{G^1(t-t'')G^1(t-t') + G^1(t-t'-t'')G^1(t)\} \\ & \sim \frac{2\Lambda^{2(\gamma_\rho+\gamma_\sigma)}}{\prod_{\nu=\rho,\sigma} u_\nu^{2\gamma_\nu+1}} \int_{-\infty}^{\infty} dt \frac{e^{i2\mu t}}{\prod_{\nu=\rho,\sigma} ((\Lambda/u_\nu) + it)^{2\gamma_\nu+1}}. \end{aligned} \quad (23)$$

An analytical expression for this integral is available [42], and given in Appendix B. The treatment of the remaining

integrals over  $t', t''$  and the calculation of the Andreev contribution is the same as in the non-interacting case and we obtain for the current  $I_1$ , in leading order in  $\mu/\Delta$  and in the small parameters  $2\Lambda\mu/u_\nu$ ,

$$I_1 = \frac{I_1^0}{\Gamma(2\gamma_\rho + 2)} \frac{v_F}{u_\rho} \left[ \frac{2\mu\Lambda}{u_\rho} \right]^{2\gamma_\rho}. \quad (24)$$

In (24) we used  $K_\sigma = 1$  and  $u_\sigma = v_F$ . The interaction suppresses the current considerably and the bias dependence has its characteristic non-linear form,  $I_1 \propto (\mu)^{2\gamma_\rho+1}$ , with an interaction dependent exponent  $2\gamma_\rho + 1$ . The parameter  $\gamma_\rho$  is the exponent for tunneling into the bulk of a single LL, i.e.  $\rho(\varepsilon) \sim |\varepsilon|^{\gamma_\rho}$  [28], where  $\rho(\varepsilon)$  is the single particle DOS. Note that the current  $I_1$  does not show a dependence on the correlation time  $1/\Delta$ , which is a measure for the time separation between the two electron tunneling-events. This is so since the two partners of a Cooper pair tunnel to *different* LL leads with no interaction-induced correlations between the leads.

### VIII. CURRENT $I_2$ FOR TUNNELING OF TWO ELECTRONS INTO THE SAME LEAD

The main new feature for the case where two electrons, originating from an Andreev process, tunnel into the same lead, is now that the 4-point correlation function of the LL no longer factorizes as was the case before when the two electrons tunnel into different leads (see (8)). In addition the two electrons will tunnel into the lead preferably from the same spatial point on the SC, i.e.  $\delta\mathbf{r} = 0$ . We denote by  $I_2$  the current for coherent transport of two electrons into the *same* lead, either lead 1 or lead 2. It can be written in a similar way as  $I_1$  (see (7)) with the difference that now we consider final states with two additional electrons (of opposite spin) in the same lead (either 1 or 2) compared to the initial state. We then obtain for  $I_2$ ,

$$I_2 = 4e \lim_{\eta \rightarrow 0} \sum_{s,s'} \int_{-\infty}^{\infty} dt \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} e^{i(2t-t'-t'')\mu} \langle H_{Tn-s'}^\dagger(t-t'') H_{Tns'}^\dagger(t) H_{Tn-s}(t') H_{Tns}(t) \rangle, \quad (25)$$

where we have used that the leads 1 and 2 are identical which results in an additional factor of two. Again, the thermal average is to be taken at  $T = 0$ , and this groundstate expectation value factorizes into a SC-part times a LL-part. However, the LL-part does not factorize anymore due to strong correlations between the two

tunneling electrons. We obtain in this case

$$\begin{aligned} & \sum_{s,s'} \langle H_{Tn-s'}^\dagger(t-t'') H_{Tns'}^\dagger(t) H_{Tn-s}(t') H_{Tns}(t) \rangle \\ &= |t_0|^4 \sum_s \langle \psi_{ns}(t-t'') \psi_{n-s}(t) \psi_{n-s}^\dagger(t') \psi_{ns}^\dagger(t) \rangle \\ & \times \langle \Psi_s^\dagger(\mathbf{r}_n, t-t'') \Psi_{-s}^\dagger(\mathbf{r}_n, t) \Psi_{-s}(\mathbf{r}_n, t') \Psi_s(\mathbf{r}_n) \rangle \\ &+ |t_0|^4 \sum_s \langle \psi_{n-s}(t-t'') \psi_{ns}(t) \psi_{n-s}^\dagger(t') \psi_{ns}^\dagger(t) \rangle \\ & \times \langle \Psi_{-s}^\dagger(\mathbf{r}_n, t-t'') \Psi_s^\dagger(\mathbf{r}_n, t) \Psi_{-s}(\mathbf{r}_n, t') \Psi_s(\mathbf{r}_n) \rangle. \end{aligned} \quad (26)$$

The 4-point correlation functions for the SC in (26) are the same as in (8) for the case when the two electrons tunnel into different leads, except that now  $\delta\mathbf{r} = 0$ . The (normalized) 4-point correlation functions in (26) for the LL are  $G_{rr',1}^2(t, t', t'') \equiv \langle \psi_{nr's}(t-t'') \psi_{nr'-s}(t) \psi_{nr'-s}^\dagger(t') \psi_{nr's}^\dagger(t) \rangle / \langle \psi_{nr's}(t-t'') \psi_{nr's}^\dagger(t) \rangle \langle \psi_{nr'-s}(t-t') \psi_{nr'-s}^\dagger(t') \rangle$ , which can be calculated using similar methods as described above for the single-particle correlation function. After some calculation we get

$$\begin{aligned} G_{rr',1}^2(t, t', t'') = & \prod_{\nu=\rho,\sigma} \left( \frac{\Lambda - iu_\nu t''}{\Lambda + iu_\nu(t-t'-t'')} \right)^{\gamma_{\nu rr'}} \left( \frac{\Lambda + iu_\nu t'}{\Lambda + iu_\nu t} \right)^{\gamma_{\nu rr'}} \\ & \times \left[ \frac{(\Lambda + iu_\sigma(t-t'-t''))(\Lambda + iu_\sigma t)}{(\Lambda + iu_\rho(t-t'-t''))(\Lambda + iu_\rho t)} \right]^{\frac{1+rr'}{4}} \\ & \times \left[ \frac{(\Lambda - iu_\rho t'')(\Lambda + iu_\rho t')}{(\Lambda - iu_\sigma t'')(\Lambda + iu_\sigma t')} \right]^{\frac{1+rr'}{4}} \end{aligned} \quad (27)$$

where  $\gamma_{\nu rr'} = \xi_\nu((1/K_\nu) + rr'K_\nu - (1+rr'))/4$  with  $\xi_\rho/\sigma = \pm 1$ . The exponent  $\gamma_{\nu rr'}$  is related to  $\gamma_\nu$ , introduced in the single-particle correlation function (13) via  $\gamma_{\nu rr'} = \xi_\nu \gamma_\nu$  for  $r = r'$ , and  $\gamma_{\nu rr'} = \xi_\nu(2\gamma_\nu + 1 - K_\nu)/2$  for  $r \neq r'$ . For the other sequence  $G_{rr',2}^2 = \langle \psi_{nr-s}(t-t'') \psi_{nr's}(t) \psi_{nr-s}^\dagger(t') \psi_{nr's}^\dagger(t) \rangle / \langle \psi_{nr-s}(t-t'') \psi_{nr-s}^\dagger(t) \rangle \langle \psi_{nr's}(t-t') \psi_{nr's}^\dagger(t') \rangle$ , we obtain

$$\begin{aligned} G_{rr',2}^2(t, t', t'') = & \prod_{\nu=\rho,\sigma} \left( \frac{\Lambda - iu_\nu t''}{\Lambda + iu_\nu(t-t'')} \right)^{\gamma_{\nu rr'}} \left( \frac{\Lambda + iu_\nu t'}{\Lambda + iu_\nu(t-t')} \right)^{\gamma_{\nu rr'}} \\ & \times \left[ \frac{(\Lambda + iu_\sigma(t-t''))(\Lambda + iu_\sigma(t-t'))}{(\Lambda + iu_\rho(t-t''))(\Lambda + iu_\rho(t-t'))} \right]^{\frac{1+rr'}{4}} \\ & \times \left[ \frac{(\Lambda - iu_\rho t'')(\Lambda + iu_\rho t')}{(\Lambda - iu_\sigma t'')(\Lambda + iu_\sigma t')} \right]^{\frac{1+rr'}{4}}. \end{aligned} \quad (28)$$

We remark that contributions from other combinations of left- and right- movers, as indicated in (27) and (28), are negligible. A contribution like  $\langle \psi_{nr's}(t-t'') \psi_{nr-s}(t) \psi_{nr'-s}^\dagger(t') \psi_{nr's}^\dagger(t) \rangle_{r \neq r'}$  is only non-zero if spin



exchange between right- and left-movers is possible, but this is a backscattering process which we explicitly ex-

clude. Using (25) together with (28), we obtain a formal expression for  $I_2$  (with  $\delta r = 0$ )

$$I_2 = 4e \left( \frac{\pi \nu_S \Delta |t_0|^2}{V} \right)^2 \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} dt \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} e^{i(2t-t'-t'')\mu} H_0^{(1)}(t''\Delta) H_0^{(2)}(t'\Delta) \\ \times \sum_{b=\pm 1} (G_{b1}^2(t, t', t'') G^1(t-t'') G^1(t-t') - G_{b2}^2(t, t', t'') G^1(t-t'-t'') G^1(t)). \quad (29)$$

In (29) the meaning of the summation index is  $b \equiv +1$  for  $rr' = +1$ , and  $b \equiv -1$  for  $rr' = -1$ . We proceed to evaluate the current  $I_2$  with  $K_\sigma = 1$  and  $u_\sigma = v_F$ , i.e.  $\gamma_{\sigma rr'} = 0$ . Since  $\gamma_{\rho rr'} > 0$ , we see from the first line in (27) and (28) that for  $|t| > \Lambda/u_\rho, t', t''$ , the full 4-point correlation function is suppressed by a factor  $\sim [t't''/(t^2)]^{\gamma_{\rho rr'}}$  compared to its factorization approximation. To calculate the current  $I_2$  we assume that the time scales  $\Lambda/u_\rho$  and  $\Lambda/v_F$  are the smallest ones in the problem. The times  $\Lambda/u_\rho$  and  $\Lambda/v_F$  are both on the order of the inverse Fermi energy in the LL, which is larger than

the energy gap  $\Delta$  and the bias  $\mu$ . By applying the same arguments as in Section V for the current  $I_1$ , we approximate the current  $I_2$ , assuming  $|t| > t', t'' > \Lambda/v_F, \Lambda/u_\rho$ , which is accurate in leading order in the small parameters  $\mu/\Delta, \Lambda\Delta/u_\nu$  and  $\Lambda\mu/u_\nu$ . In this limit we obtain for the two-particle correlation functions  $G_{rr'1}^2 = -G_{rr'2}^2 = u_\rho^{2\gamma_{\rho rr'}} (t't'')^{\gamma_{\rho rr'}} / (\Lambda + iu_\rho t)^{2\gamma_{\rho rr'}}$ . The current  $I_2$  for tunneling of two electrons into the *same* lead 1 or 2 then becomes (for  $\delta r = 0$ )

$$I_2 = 2e \left( \frac{2\pi \nu_S \Delta |t_0|^2}{V} \right)^2 \frac{\Lambda^{2\gamma_\rho}}{u_\rho^{2\gamma_\rho+1} v_F} \lim_{\eta \rightarrow 0} \sum_{b=\pm 1} \int_0^{\infty} dt' \int_0^{\infty} dt'' e^{-\eta(t'+t'')} (t't'')^{\gamma_{\rho b}} H_0^{(1)}(t''\Delta) H_0^{(2)}(t'\Delta) \\ \times \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \frac{e^{i2\mu t}}{\left(\frac{\Lambda}{u_\rho} + it\right)^{2\gamma_{\rho b}+2\gamma_\rho+1} \left(\frac{\Lambda}{v_F} + it\right)}. \quad (30)$$

Again, the integrals appearing in (30) can be evaluated analytically [42] with the results given in Appendix B. Note that according to the two-particle LL-correlation functions (27) and (28), we find that the dynamics coming from the delay times  $t'$  and  $t''$  cannot be neglected anymore, as was done in [32]. We evaluate (30) in leading order in  $2\mu\Lambda/u_\nu$  and finally obtain for the current  $I_2$

$$I_2 = I_1 \sum_{b=\pm 1} A_b \left( \frac{2\mu}{\Delta} \right)^{2\gamma_{\rho b}}. \quad (31)$$

The interaction dependent constant  $A_b$  in (31) is given by

$$A_b = \frac{2^{2\gamma_{\rho b}-1}}{\pi^2} \frac{\Gamma(2\gamma_\rho+2)}{\Gamma(2\gamma_{\rho b}+2\gamma_\rho+2)} \Gamma^4 \left( \frac{\gamma_{\rho b}+1}{2} \right), \quad (32)$$

which is decreasing for increasing the interactions in the LL leads and  $\Gamma(x)$  is the Gamma function. We remark

that in (31) the current  $I_1$  is to be taken at  $\delta r = 0$ . The non-interacting limit,  $I_2 = I_1 = I_1^0$ , is recovered by putting  $\gamma_\rho = \gamma_{\rho b} = 0$ , and  $u_\rho = v_F$ . The result for  $I_2$  shows that the unwanted injection of two electrons into the same lead is suppressed compared to  $I_1$  by a factor of  $A_+(2\mu/\Delta)^{2\gamma_{\rho+}}$  if both electrons are injected into the same branch (left or right movers), or by  $A_-(2\mu/\Delta)^{2\gamma_{\rho-}}$  if the two electrons travel in different directions. Since it holds that  $\gamma_{\rho-} = \gamma_{\rho+} + (1 - K_\rho)/2 > \gamma_{\rho+}$ , it is more favorable that two electrons travel in the same direction than in opposite directions. The suppression of the current  $I_2$  by  $1/\Delta$  shows very nicely the two-particle correlation effect for the coherent tunneling of two electrons into the same lead. The larger  $\Delta$ , the shorter is the delay time between the arrivals of the two partner electrons of a given Cooper pair, and, in turn, the more the second electron will be influenced by the presence of the first one already in the LL. By increasing the bias  $\mu$  the electrons can tunnel faster through the barrier due to more

channels becoming available into which the electron can tunnel, and therefore the effect of  $\Delta$  is less pronounced. Also note that this correlation effect disappears when interactions are absent in the LL ( $\gamma_\rho = \gamma_{\rho b} = 0$ ).

## IX. EFFICIENCY AND DISCUSSION

We have established now that there exists indeed the suppression for tunneling of two spin-entangled electrons into the same LL-lead compared to the desired process where the two electrons tunnel into different leads. However, we have to take into account that the process into different leads suffers also a suppression due to a finite tunneling separation  $\delta r$  of the two electrons forming a Cooper pair in the SC. In Section VI we showed that this suppression can be considerably reduced if one uses effectively low-dimensional SCs. To estimate the efficiency of the entangler we form the ratio  $I_1/I_2$  and demand that it is larger than one. This requirement is fulfilled if approximately

$$A_+ \left( \frac{2\mu}{\Delta} \right)^{2\gamma_{\rho+}} < 1/(k_F \delta r)^{d-1}, \quad (33)$$

where  $d$  is the dimension of the SC, and it is assumed that the coherence length  $\xi$  of the SC is large compared to  $\delta r$ . The leading term of  $I_2$  is proportional to  $(2\mu/\Delta)^{2\gamma_{\rho+}}$  describing the power-law suppression, with exponent  $2\gamma_{\rho+} = 2\gamma_\rho$ , of the process where two electrons, entering the same lead, will propagate in the same direction. The exponent  $\gamma_{\rho+}$  is the exponent for the single-particle tunneling-DOS from a metal (SC) into the center (bulk) of a LL. Experimentally accessible systems which exhibit LL-behavior are metallic carbon nanotubes. It was pointed out [43, 44] that the long range part of the Coulomb interaction, which is dominated by forward scattering events with small momentum transfer, can lead to LL behavior in carbon nanotubes with very small values of  $K_\rho \sim 0.2 - 0.3$ , as measured experimentally [24, 45] and predicted theoretically [44]. This would correspond to an exponent  $2\gamma_\rho \sim 0.8 - 1.6$ , which seems very promising. In addition, single-wall nanotubes show similar tunneling exponents as derived here. The tunneling DOS for a single-wall nanotube is predicted to be  $\rho(\varepsilon) \sim |\varepsilon|^\eta$  with  $\eta = (K_\rho^{-1} + K_\rho - 2)/8$  [43, 44], which is half of  $\gamma_\rho$ , and was measured [24, 45] to be  $\sim 0.3 - 0.4$ . Similar values were found also in multiwall-nanotubes [46]. It is known that the power-law suppression of the single-particle DOS is even larger if one considers tunneling into the *end* of a LL. For single wall nanotubes one finds  $\eta_{\text{end}} = (K_\rho^{-1} - 1)/4 > \eta$  [43, 44], or for conventional LL-theory again an enhancement by a factor of two [47]. We therefore expect to get an even stronger suppression if the Cooper pairs tunnel into the end of the LLs. We remark that the non-locality of the two electrons could be probed via the Aharonov-Bohm oscillations in the current, when the leads 1,2 are formed into a loop enclosing a magnetic flux. Due to the different

paths which the electrons can choose to go around the loop, we expect to see  $h/e$  and  $h/2e$  oscillation periods, as a function of magnetic flux, in the current like for non-interacting leads [16]. Interference of contributions where the two electrons travel through different leads with contributions where they travel through the same lead then lead to the  $h/e$  oscillations, whereas interference of contributions where both electrons travel through the same arm 1 or 2 of the loop lead to the  $h/2e$  oscillations. The amplitudes of these oscillations must be related to the currents describing the interfering processes. We expect that the  $h/e$  oscillation contribution should be  $\propto (I_1 I_2)^\alpha$  and the  $h/2e$  oscillation contribution should be  $\propto I_2^{2\alpha}$ , with an exponent  $\alpha$  that has to be determined by explicit calculations. In the non-interacting limit  $\alpha$  should be  $1/2$  [16]. The different periods then allow for an experimental test of how successful the separation of the two electrons is. For instance, if the two electrons only can tunnel into the same lead, e.g. if  $k_F \delta r$  is too large or the interaction in the leads too weak, then  $I_1 \sim 0$  and we would only see the  $h/2e$  oscillations in the current. The determination of the precise value of the exponents will be deferred to another publication since it requires a separate calculation including finite size properties of the LL along the lines discussed in [51].

## X. DECAY OF THE ELECTRON-SINGLET DUE TO LL-INTERACTIONS

We have shown in the preceding sections that the interaction in a LL-lead can help to separate two spin-entangled electrons so that the two electrons enter *different* leads. A natural question then arises: what is the lifetime of a (non-local) spin-singlet state formed of two electrons which are injected into different LL-leads, one electron per lead? To address this issue we introduce the following correlation function

$$P(\mathbf{r}, t) = |\langle S(\mathbf{r}, t) | S(0, 0) \rangle|^2. \quad (34)$$

This function is the probability density that a singlet state, injected at point  $\mathbf{r} \equiv (x_1, x_2) = 0$  and at time  $t = 0$ , is found at some later time  $t$  and at point  $\mathbf{r}$ . Therefore,  $P(\mathbf{r}, t)$  is a measure of how much of the initial singlet state remains after the two injected electrons have interacted with all the other electrons in the LL during the time interval  $t$ . Here

$$|S(\mathbf{r}, t)\rangle = \sqrt{\pi\alpha} [\psi_{1\uparrow}^\dagger(x_1, t)\psi_{2\downarrow}^\dagger(x_2, t) - \psi_{1\downarrow}^\dagger(x_1, t)\psi_{2\uparrow}^\dagger(x_2, t)]|0\rangle \quad (35)$$

is the electron singlet state created on top of the LL groundstates. The extra normalization factor  $\sqrt{2\pi\alpha}$  is introduced to guarantee  $\int d\mathbf{r} P(\mathbf{r}, t) = 1$  in the non-interacting limit and corresponds to the replacement of  $\psi_{ns}$  by  $(2\pi\alpha)^{1/4}\psi_{ns}$ . The singlet-singlet correlation function factorizes into two single-particle Green's functions due to negligible interaction between the leads 1 and 2. Therefore we

have  $P(\mathbf{r}, t) = (2\pi\alpha)^2 \prod_n |\langle \psi_{ns}(x_n, t) \psi_{ns}^\dagger(0, 0) \rangle|^2$ , with  $\langle \psi_{ns}(x_n, t) \psi_{ns}^\dagger(0, 0) \rangle = \sum_{r=\pm} e^{ik_F r x_n} G_{nrs}^1(x_n, t)$ . For simplicity we just study the slow spatial variations of  $|\langle \psi_{ns}(x_n, t) \psi_{ns}^\dagger(0, 0) \rangle|^2$  and obtain with (13)

$$(2\pi)^2 |\langle \psi_{ns}(x_n, t) \psi_{ns}^\dagger(0, 0) \rangle|^2 = \sum_{r=\pm} \lim_{\alpha \rightarrow 0} \frac{\Lambda^2 + (v_F t - r x_n)^2}{\alpha^2 + (v_F t - r x_n)^2} \prod_{\nu=\rho, \sigma} \frac{1}{\sqrt{\Lambda^2 + (r x_n - u_\nu t)^2}} \times \left( \frac{\Lambda^4}{(\Lambda^2 + x_n^2 + (u_\nu t)^2)^2 + (2u_\nu t \Lambda)^2} \right)^{\gamma_\nu/2}. \quad (36)$$

If we use  $\pi\delta(x) = \lim_{\alpha \rightarrow 0} \alpha/(\alpha^2 + x^2)$  we can then write the remaining probability of the singlet as

$$P(\mathbf{r}, t) = \prod_n \frac{1}{2} \sum_{r=\pm} F(t) \delta(x_n - r v_F t) \quad (37)$$

with a time decaying weight factor of the  $\delta$ -function

$$F(t) = \prod_{\nu=\rho, \sigma} \sqrt{\frac{\Lambda^2}{\Lambda^2 + (v_F - u_\nu)^2 t^2}} \times \left( \frac{\Lambda^4}{(\Lambda^2 + (v_F t)^2 + (u_\nu t)^2)^2 + (2\Lambda u_\nu t)^2} \right)^{\gamma_\nu/2} \quad (38)$$

Without interaction we have  $F(t) = 1$ , which means that there is no decay of the singlet state. As interactions are turned on, we see that for times  $t > \Lambda/u_\nu$  the singlet state decays in time with approximately  $F(t) \sim \prod_{\nu=\rho, \sigma} \frac{\Lambda}{\sqrt{\Lambda^2 + (u_\nu - v_F)^2 t^2}} \left( \frac{\Lambda}{t \sqrt{v_F^2 + u_\nu^2}} \right)^{2\gamma_\nu}$ . This result together with (37) shows that charge and spin of an electron propagate with velocity  $v_F$ , whereas charge (spin) excitations of the LL propagate with  $u_\rho$  ( $u_\sigma$ ). In addition, we see that the probability  $P(\mathbf{r}, t)$  shows an additional power-law decay  $\sim (\frac{\Lambda}{t \sqrt{v_F^2 + u_\nu^2}})^{2\gamma_\nu}$ , with an interaction dependent exponent. We will show in the next section that although the singlet gets destroyed due to interactions, we still can observe charge and spin of the initial singlet via the spin and charge density fluctuations of the LL.

## XI. PROPAGATION OF CHARGE AND SPIN

The charge and spin propagation as a function of time in a state  $|\Psi\rangle$  can be described by the correlation function  $\langle \Psi | \rho(x, t) | \Psi \rangle$  for the charge, and  $\langle \Psi | \sigma_z(x, t) | \Psi \rangle$  for the spin. The normal-ordered charge density operator for LL  $n$  is  $\rho_n(x_n) = \sum_s : \psi_{ns}^\dagger(x_n) \psi_{ns}(x_n) := \sum_{sr} : \psi_{nsr}^\dagger(x_n) \psi_{nsr}(x_n) :$ , if we only consider the slow spatial variations of the density operator. Similarly, the normal-ordered spin density operator in  $z$ -direction is  $\sigma_n^z(x_n) = \sum_{sr} s : \psi_{nsr}^\dagger(x_n) \psi_{nsr}(x_n) :$ . These density

fluctuations can be expressed in a bosonic form (see Appendix A) as

$$\rho_n(x_n) = \frac{\sqrt{2}}{\pi} \partial_x \phi_{n\rho}(x_n) \quad (39)$$

and for the spin

$$\sigma_n^z(x_n) = \frac{\sqrt{2}}{\pi} \partial_x \phi_{n\sigma}(x_n). \quad (40)$$

We now consider a state  $|\Psi\rangle = \psi_{nsr}^\dagger(x_n)|0\rangle$  where we inject an electron at time  $t = 0$  into branch  $r$  on top of the LL groundstate in lead  $n$  and calculate the time dependent charge and spin density fluctuations according to  $\langle 0 | \psi_{nsr}(x_n) \rho_n(x'_n, t) \psi_{nsr}^\dagger(x_n) | 0 \rangle$  for the charge and similar for the spin  $\langle 0 | \psi_{nsr}(x_n) \sigma_n^z(x'_n, t) \psi_{nsr}^\dagger(x_n) | 0 \rangle$ . If we express the bosonic field operators  $\phi_{n\nu}$  and  $\theta_{n\nu}$  in terms of the boson modes shown in (10) and (11) and the Fermi operators according to the bosonization dictionary (4) we obtain for the charge fluctuations

$$\begin{aligned} & \langle 0 | \psi_{nsr}(x_n) \rho_n(x'_n, t) \psi_{nsr}^\dagger(x_n) | 0 \rangle \\ &= \frac{1}{2} (1 + r K_\rho) \delta(x'_n - x_n - u_\rho t) \\ &+ \frac{1}{2} (1 - r K_\rho) \delta(x'_n - x_n + u_\rho t), \end{aligned} \quad (41)$$

and for the spin fluctuations

$$\begin{aligned} & \langle 0 | \psi_{nsr}(x_n) \sigma_n^z(x'_n, t) \psi_{nsr}^\dagger(x_n) | 0 \rangle \\ &= \frac{s}{2} (1 + r K_\sigma) \delta(x'_n - x_n - u_\sigma t) \\ &+ \frac{s}{2} (1 - r K_\sigma) \delta(x'_n - x_n + u_\sigma t). \end{aligned} \quad (42)$$

The results (41) and (42) are obtained by sending  $\Lambda \rightarrow 0$  and by using the same normalization convention for the electron operators as in (35). We see that in contrast to the singlet, the charge and spin density fluctuations in the LL created by the injected electron do not decay and show a pulse shape with no dispersion in time. This is due to the linear energy dispersion relation of the LL-model. In carbon nanotubes such a highly linear dispersion relation is indeed realized, and, therefore, nanotubes should be well suited for spin transport. Another interesting effect that shows up in (41) and (42) is the different velocities of spin and charge, which is known as spin-charge separation. It would be interesting to test Bell inequalities [4] via spin-spin correlation measurements between the two LL-leads and see if the initial entanglement of the spin singlet is still observable in the spin density-fluctuations. Although detection of single spins with magnitudes on the order of electron spins has still not been achieved, magnetic resonance force microscopy (MFRM) seems to be very promising in doing so [48]. Another scenario is to use the LL just as an intermediate medium which is needed to first separate the two electrons of a Cooper pair and then to take them (in general other electrons) out again into two (spatially separated)

Fermi liquid leads where the (possibly reduced) spin entanglement could be measured via the current noise in a beamsplitter experiment [18]. Similarly, to test Bell inequalities one can make then use of measuring spin via the charge of the electron [9, 19, 20]. In this context we finally note that the decay of the singlet state given by (37) sets in almost immediately after the injection into the LLs (the time scale is approximately the inverse of the Fermi energy), but at least at zero temperature, the suppression is only polynomial in time, which suggests that some fraction of the singlet state can still be recovered.

## XII. CONCLUSIONS

We proposed an s-wave superconductor (SC), coupled to two spatially separated Luttinger liquid (LL) leads, as an entangler for electron spins. We showed that the strong correlations present in the LL can be used to separate two electrons, forming a spin-singlet state, which originate from an Andreev tunneling process of a Cooper pair from the SC to the leads. We have shown that the coherent tunneling of two electrons into the same lead is suppressed by a characteristic power law in the small parameter  $\mu/\Delta$ , where  $\mu$  is the applied bias between the SC and the LL-leads, and  $\Delta$  is the gap in the SC. On the other hand, when the two electrons tunnel into different leads, the current is suppressed by the initial separation of the two electrons. This suppression, however, can be considerably reduced by going over to effective lower-dimensional SC. We also addressed the question of how much of the initial singlet can be taken out of the LL at some later time, and we found that the probability is decreasing in time, again with a power-law (at zero temperature). Nevertheless, the spin information can still be transported through the wires by means of the (proper) spin excitations of the LL.

While preparing this manuscript we have learned of related and independent efforts by S. Vishveshwara et al.[49] who consider a similar setup as proposed here thereby arriving at similar conclusions.

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## APPENDIX A: FINITE SIZE DIAGONALIZATION OF THE LL-HAMILTONIAN

In this Appendix we derive the diagonalized form of the LL-Hamiltonian (3) including terms of order  $1/L$ , which describe integer charge and spin excitations. For simplicity, we consider only one LL and will therefore suppress the subscript  $n$  for the leads. We start with the exact bosonization dictionary for the Fermi-operator for electrons on branch  $r = \pm$  [29, 30],

$$\begin{aligned} \psi_{rs}(x) = & \lim_{\alpha \rightarrow 0} \frac{U_{r,s}}{\sqrt{2\pi\alpha}} \exp \left\{ ir(p_F - \pi/L)x \right. \\ & \left. + \frac{ir}{\sqrt{2}} (\phi_\rho(x) + s\phi_\sigma(x) - r(\theta_\rho(x) + s\theta_\sigma(x))) \right\}. \end{aligned} \quad (\text{A1})$$

The  $U_{r,s}$ -operator (often denoted as Klein factor) is unitary and decreases the number of electrons with spin  $s$  on branch  $r$  by one. This operator also ensures the correct anticommutation relations for  $\psi_{rs}(x)$ . The normal ordered charge density operator is  $\rho(x) = \sum_{sr} : \psi_{rs}^\dagger(x) \psi_{rs}(x) :$ , where  $: :$  measures the corresponding quantity relative to the groundstate with all single particle states filled up to the chemical potential  $\mu_l$ . The normal ordered spin density operator is defined by  $\sigma^z(x) = \sum_{sr} s : \psi_{rs}^\dagger(x) \psi_{rs}(x) :$ . In addition, one can define (bare) current density operators for charge  $j_\rho = \sum_{sr} r \psi_{rs}^\dagger(x) \psi_{rs}(x)$ , and for the spin  $j_\sigma = \sum_{sr} rs \psi_{rs}^\dagger(x) \psi_{rs}(x)$ , respectively. Note that the current density has not to be normal ordered since its groundstate expectation value vanishes. The normal ordered product  $: \psi_{rs}^\dagger(x) \psi_{rs}(x) :$  is calculated according to

$$: \psi_{rs}^\dagger(x) \psi_{rs}(x) := \lim_{\Delta x \rightarrow 0} : \psi_{rs}^\dagger(x + \Delta x) \psi_{rs}(x) :. \quad (\text{A2})$$

By expanding the operator product in (A2) within the normal-order sign, the right-hand side of (A2) equals  $(1/2\pi) \partial_x (\phi_\rho(x) + s\phi_\sigma(x) - r(\theta_\rho(x) + s\theta_\sigma(x))) / \sqrt{2}$ , from which one easily finds

$$\rho(x) = \frac{\sqrt{2}}{\pi} \partial_x \phi_\rho(x), \quad \sigma^z(x) = \frac{\sqrt{2}}{\pi} \partial_x \phi_\sigma(x), \quad (\text{A3})$$

and for the current densities

$$j_\rho(x) = -\sqrt{2} \Pi_\rho(x), \quad j_\sigma(x) = -\sqrt{2} \Pi_\sigma(x). \quad (\text{A4})$$

The field  $\Pi_\nu(x)$  is related to  $\theta_\nu(x)$  by  $\partial_x \theta_\nu(x) = \pi \Pi_\nu(x)$ . We decompose the phase fields into  $\phi_\nu(x) = \phi_\nu^P(x) + \phi_\nu^0(x)$  and  $\Pi_\nu(x) = \Pi_\nu^P(x) + \Pi_\nu^0(x)$ , where the part with non-zero momentum  $\phi_\nu^P$  and  $\Pi_\nu^P$  can be expanded in a series of normal modes

$$\phi_\nu^P(x) = \frac{1}{\sqrt{L}} \sum_{p \neq 0} \frac{1}{\sqrt{2\omega_{p\nu}}} e^{ipx} e^{-\alpha|p|/2} (b_{\nu p} + b_{\nu -p}^\dagger), \quad (\text{A5})$$

and for the canonical momentum

$$\Pi_\nu^P(x) = \frac{-i}{\sqrt{L}} \sum_{p \neq 0} \sqrt{\frac{\omega_{p\nu}}{2}} e^{ipx} e^{-\alpha|p|/2} (b_{\nu p} - b_{\nu-p}^\dagger). \quad (\text{A6})$$

These fields have to satisfy bosonic commutation relations  $[\phi_\nu^P(x), \Pi_\mu^P(x')] = i\delta_{\nu\mu}(\delta(x-x') - 1/L)$ , which in turn demands  $[b_{\nu p}, b_{\mu p'}^\dagger] = \delta_{\nu\mu}\delta_{pp'}$  and  $[b_{\nu p}, b_{\mu p'}] = [b_{\nu p}^\dagger, b_{\mu p'}^\dagger] = 0$ . The zero mode parts  $\phi_\nu^0$  and  $\Pi_\nu^0$  can be found by considering the integrated charge (spin) and charge- (spin)-currents, respectively. For instance the integrated charge density  $\sum_{rs} N_{rs} = \int dx \rho(x) = (\sqrt{2}/\pi)(\phi_\rho(L/2) - \phi_\rho(-L/2)) = (\sqrt{2}/\pi)(\phi_\rho^0(L/2) - \phi_\rho^0(-L/2))$ . Similar results are obtained for the other density operators, which then implies the zero modes to be  $\phi_\nu^0(x) = (\pi/L)(N_{+\nu} + N_{-,\nu})x$  and  $\Pi_\nu^0 = -(1/L)(N_{+\nu} - N_{-,\nu})$ , where  $N_{r\rho/\sigma} = (N_{r\uparrow} \pm N_{r\downarrow})/\sqrt{2}$ . The LL-Hamiltonian (3) is then diagonalized by the following expansion of the bosonic fields

$$\begin{aligned} \phi_\nu(x) &= \sum_{p \neq 0} \sqrt{\frac{\pi K_\nu}{2|p|L}} e^{ipx} e^{-\alpha|p|/2} (b_{\nu p} + b_{\nu-p}^\dagger) \\ &+ \frac{\pi}{L}(N_{+\nu} + N_{-,\nu})x, \end{aligned} \quad (\text{A7})$$

and for the canonical conjugate momentum operator,

$$\begin{aligned} \Pi_\nu(x) &= -i \sum_{p \neq 0} \sqrt{\frac{|p|}{2\pi L K_\nu}} e^{ipx} e^{-\alpha|p|/2} (b_{\nu p} - b_{\nu-p}^\dagger) \\ &- \frac{1}{L}(N_{+\nu} - N_{-,\nu}), \end{aligned} \quad (\text{A8})$$

where we have used  $\omega_{p\nu} = |p|/K_\nu\pi$ . For the operator  $K_L = H_L - \mu_l N$  we then obtain

$$\begin{aligned} K_L &= \sum_{p \neq 0, \nu} u_\nu |p| b_{p\nu}^\dagger b_{p\nu} \\ &+ \frac{2\pi}{L} \left[ \frac{u_\nu}{K_\nu} (N_{+\nu} + N_{-,\nu})^2 + u_\nu K_\nu (N_{+\nu} - N_{-,\nu})^2 \right]. \end{aligned} \quad (\text{A9})$$

In (A9) we have subtracted the zero point energy  $(1/2) \sum_{p \neq 0, \nu} u_\nu |p|$ , which originates from an infinite filled Dirac sea of negative energy particle states in the LL-model. The zero modes in (A7) and (A8) give rise to contributions of order  $1/L$  in the Hamiltonian (A9), and they are also responsible for a shift of the Fermi wavevector  $p_F$ , appearing in the Fermionic field operator  $\psi_{ns}(x)$ , by a contribution of the same order. Since we are only interested in the thermodynamic limit, we have neglected the zero mode contributions in explicit calculations in the main text.

## APPENDIX B: EXACT RESULTS FOR THE TIME INTEGRALS

In this Appendix we give the exact results for the time integrals in (23) and (30). The integrals over the time variable  $t$  appearing in (23) and (30) have the form

$$\int_{-\infty}^{\infty} dt \frac{e^{i2\mu t}}{\left(\frac{\Lambda}{u_\rho} + it\right)^Q \left(\frac{\Lambda}{u_\sigma} + it\right)^R} = \frac{2\pi e^{-2\mu \frac{\Lambda}{u_\rho}} (2\mu)^{Q+R-1}}{\Gamma(Q+R)} {}_1F_1(R; Q+R; 2\Lambda(u_\rho^{-1} - u_\sigma^{-1})\mu), \quad (\text{B1})$$

with  $Q, R \geq 1$ , but in general this integral is valid for  $Q, R$  satisfying  $\text{Re}(Q+R) > 1$  (see [42] p. 345). The function  $\Gamma(x)$  in (B1) is the Gammafunction and  ${}_1F_1(\alpha; \gamma; z)$  is the confluent hypergeometric function given by

$${}_1F_1(\alpha; \gamma; z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \dots \quad (\text{B2})$$

In the main text we considered only the leading order term of  ${}_1F_1$  since higher order terms are small by the pa-

rameters  $2\mu\Lambda/u_\rho$  and  $2\mu\Lambda/u_\sigma$ . The integrals over the delay times  $t'$  and  $t''$  in (30) contain Hankelfunctions of the first and second kind which are linear combinations of Besselfunctions of the first and second kind, i.e.  $H_0^{(1/2)}(t\Delta) = J_0(t\Delta) \pm iY_0(t\Delta)$ . The integrals over  $t'$  and  $t''$  in (30) are therefore linear combinations of integrals of the form

$$\lim_{\eta \rightarrow 0} \int_0^\infty dt e^{-\eta t} Y_0(t\Delta) t^\delta = \lim_{\eta \rightarrow 0} \left( -\frac{2}{\pi} \Gamma(\delta+1) (\Delta^2 + \eta^2)^{-\frac{1}{2}(\delta+1)} Q_\delta \left( \eta / \sqrt{\eta^2 + \Delta^2} \right) \right), \quad (\text{B3})$$

and

$$\lim_{\eta \rightarrow 0} \int_0^\infty dt e^{-\eta t} J_0(t\Delta) t^\delta = \lim_{\eta \rightarrow 0} \left( \Gamma(\delta+1) (\Delta^2 + \eta^2)^{-\frac{1}{2}(\delta+1)} P_\delta \left( \eta / \sqrt{\eta^2 + \Delta^2} \right) \right). \quad (\text{B4})$$

This result is valid for  $\delta > -1$  (see [42] p. 691). The functions  $Q$  and  $P$  are Legendre functions. The limit  $\eta \rightarrow 0$  for  $Q_\delta(\eta/\sqrt{\eta^2 + \Delta^2})$  is (see [42] p. 959)

$$Q_\delta(0) = -\frac{1}{2} \sqrt{\pi} \sin(\delta\pi/2) \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2}+1)}, \quad (\text{B5})$$

and the limit  $\eta \rightarrow 0$  for  $P_\delta(\eta/\sqrt{\eta^2 + \Delta^2})$  is

$$\begin{aligned} P_\delta(0) &= \frac{\sqrt{\pi}}{\Gamma(\frac{\delta}{2}+1) \Gamma(\frac{1-\delta}{2})} \\ &= \frac{1}{\sqrt{\pi}} \cos(\delta\pi/2) \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2}+1)}. \end{aligned} \quad (\text{B6})$$

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- [1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
  - [2] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), p. 175.
  - [3] C.H. Bennett and D. P. DiVincenzo, Nature **404**, 247 (2000).
  - [4] J.S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
  - [5] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
  - [6] K. Mattle, H. Weinfurter, P. G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. **76**, 4656 (1996).
  - [7] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature **390**, 575 (1997).
  - [8] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Phys. Rev. Lett. **80**, 1121 (1998).
  - [9] D. Loss and D.P. DiVincenzo, Phys. Rev. A **57**, 120 (1998); cond-mat/9701055.
  - [10] G. Burkard, H.-A. Engel, D. Loss, Fortschr. Phys. **48**, 965 (2000); cond-mat/0004182.
  - [11] J.M. Kikkawa, I.P. Smorchkova, N. Samarth, and D.D. Awschalom, Science **277**, 1284 (1997).
  - [12] J.M. Kikkawa and D.D. Awschalom, Phys. Rev. Lett. **80**, 4313 (1998).
  - [13] D.D. Awschalom and J.M. Kikkawa, Phys. Today **52**(6), 33 (1999).
  - [14] D.P. DiVincenzo and D. Loss, J. Magn. Magn. Mater. **200**, 202 (1999); cond-mat/9901137.
  - [15] That superconducting correlations can induce non-local spin-entanglement was pointed out already in [50]. There two superconductors, held at the same chemical potential are tunnel-coupled to two spatially separated quantum dots containing one electron-spin each and which are not tunnel-coupled among themselves. The tunnel-coupling to the superconductors then induce an effective Heisenberg spin interaction between the two spins on the dots.
  - [16] P. Recher, E.V. Sukhorukov, and D. Loss, Phys. Rev. B **63**, 165314 (2001); cond-mat/0009452.
  - [17] L.P. Kouwenhoven, G. Schön, L. L. Sohn, Mesoscopic Electron Transport, NATO ASI Series E: Applied Sciences-Vol.345, 1997, Kluwer Academic Publishers, Amsterdam.
  - [18] G. Burkard, D. Loss, and E. V. Sukhorukov, Phys. Rev. B **61**, R16 303 (2000).
  - [19] P. Recher, E.V. Sukhorukov, and D. Loss, Phys. Rev. Lett. **85**, 1962 (2000).
  - [20] H.-A. Engel and D. Loss, Phys. Rev. Lett. **86**, 4648

- (2001).
- [21] G.B. Lesovik, T. Martin, and G. Blatter, cond-mat/0009193.
  - [22] G. Falci, D. Feinberg, F.W.J. Hekking, cond-mat/0011339.
  - [23] R. Mélin, cond-mat/0105073.
  - [24] M. Bockrath *et al.*, *Nature* **397**, 598 (1999).
  - [25] M.R. Geller and D. Loss, Phys. Rev. B **62**, R16 298 (2000).
  - [26] J.R. Schrieffer, *Theory of Superconductivity*, Benjamin/Cummings, New York, 1964.
  - [27] In context with nanotubes we remark that it was pointed out [43, 44] that for the (metallic) armchair nanotubes  $(N, N)$  with  $N \gtrsim 10$  it is appropriate to use a Luttinger liquid model where backscattering and umklapp scattering can be neglected. This is so because for large  $N$ , the probability for two electrons to be near each other is small ( $\sim 1/N$ ) and therefore one can neglect the short range part  $r \sim a$ , with  $a$  the lattice spacing, of the Coulomb potential which corresponds to large momentum transfers  $\sim 2p_F$  at which backscattering and umklapp scattering takes place.
  - [28] H.J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).
  - [29] F.D.M. Haldane, J. Phys. C. **14**, 2585 (1981).
  - [30] R. Heidenreich, R. Seiler, and A. Uhlenbrock, J. Stat. Phys. **22**, 27 (1980).
  - [31] F.W.J. Hekking, L.I. Glazman, K.A. Matveev, and R.I. Shekhter, Phys. Rev. Lett. **70**, 4138 (1993).
  - [32] R. Fazio, K.H. Wagenblast, C. Winkelholz, and G. Schön, Physica B, **222**, 364 (1996).
  - [33] E. Merzbacher, *Quantum Mechanics* 3rd ed., John Wiley and Sons, New York, 1998, ch. 20.
  - [34] G.D. Mahan, *Many-Particle-Physics, third edition*, p. 563, Kluwer Academic/Plenum Publishers, New York, 2000.
  - [35] J. Sólyom, Adv. Phys. **28**, 201 (1979)
  - [36] Y. Suzumura, Prog. Theor. Phys. **63**, 5 (1980).
  - [37] H.J. Schulz, J. Phys. C **16**, 6769 (1983).
  - [38] J. Voit, J. Phys. CM **5**, 8305 (1993).
  - [39] P. Recher and D. Loss, unpublished
  - [40] A.F. Volkov, P.H.C. Magne, B.J. van Wees, and T.M. Klapwijk, Physica C **242**, 261 (1995).
  - [41] M. Kociak, A. Yu. Kasumov, S. Guron, B. Reulet, I.I. Khodos, Yu.B. Gorbatov, V.T. Volkov, L. Vaccarini, and H. Bouchiat, Phys. Rev. Lett. **86**, 2416 (2001).
  - [42] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products, Sixth Edition*, Academic Press, San Diego, 2000.
  - [43] R. Egger and A. Gogolin, Phys. Rev. Lett. **79**, 5082 (1997); R. Egger, Phys. Rev. Lett. **83**, 5547 (1999).
  - [44] C. Kane, L. Balents, and M.P. Fisher, Phys. Rev. Lett. **79**, 5086 (1997).
  - [45] J. Nygård, Ph.D. Thesis, Faculty of Science University of Copenhagen (2000).
  - [46] A. Bachthold, M. de Jonge, K. Grove-Rasmussen, and P.L. McEuen, M. Buitelaar and C. Schönenberger, Phys. Rev. Lett. **87**, 166801 (2001).
  - [47] L. Balents and R. Egger, Phys. Rev. B **64** 035310 (2001)
  - [48] D. Rugar, B.C. Stipe, H.J. Mamin, C.S. Yannoni, T.D. Stowe, K.Y. Yasumura, T.W. Kenny, Appl. Phys. A 72 [Suppl.], S3-S10 (2001).
  - [49] S. Vishveshwara, C. Bena, L. Balents, and M.P.A. Fisher, unpublished.
  - [50] M.-S. Choi, C. Bruder, and D. Loss, Phys. Rev. B **62**, 13569 (2000); cond-mat/0001011; and in Lecture Notes in Physics, Vol. 579, eds. R. Haug and H. Schoeller, Springer, Berlin, 2001, pp. 46-66.
  - [51] M. R. Geller and D. Loss. Phys. Rev. B **56** 9692 (1997).